Computational Stochastic Mechanics, Spanos (ed.) © 1995 Balkema, Rotterdam, ISBN 90 5410 527 5

Phase space flux ratio as a measure of relative stability

M. Frey
Department of Mathematics, Bucknell University, Lewisburg, Pa., USA
E. Simiu
Building and Fire Research Laboratory, National Institute of Standards and Technology, Gaithersburg, Md., USA

Abstract: A new measure of the relative stability of potential wells is proposed based on phase space transport. This measure is described for continuous one-dimensional bistable dynamical systems and contrasted with a measure of relative stability based on the stationary distribution of system state in phase space. The advantages and limitations of the proposed approach to relative stability are discussed and a "blowtorch" theorem is presented.

1. Introduction

Relative stability is a central issue in the design and analysis of many types of dynamical systems. Depending on the application, different conceptions of relative stability are employed based variously on Lyaponov exponents (Ariaratnam 1993), probability ratios (van Kampen 1988) and relative equilibria (Maddocks 1991). For systems modeled as autonomous flows at least three definitions of relative stability can be identified: linear stability, Lyapunov stability and spectral stability (Howard 1990). Stability may be interpreted in terms of potential energy; a system is considered to be in a stable state if its potential energy is at a relative minimum. If this minimum is at the bottom of a deep potential well, then the state is considered to be highly stable since strong forcing external to the system is needed to drive the system out of the well away from this stable state.

In this paper we propose a new definition of relative stability for potential wells based on phase space transport in dynamical systems. For purpose of comparison we also consider a standard definition of relative stability based on probability ratios. Our presentation is restricted to one-dimensional bistable dynamical systems and we consider two specific families of potentials.

One-dimensional Newtonian systems are dynamical systems of the form $\ddot{x} = -V'(x)$ where $x = x(t)$ is the state of the system at time t and $V(x)$ is the potential energy of the system when in state x. The double square well potential and the asymmetric Duffing potential (Brunsden 1989) are two examples of potentials of bistable systems. The potential energy $V(x)$ of the double square well system is

$$V(x) = \begin{cases} 0, & x = 0 \\ -d_1, & -w < x < 0 \\ -d_2, & 0 < x < w \\ \infty, & x \geq w \\ \infty, & x \leq -w \end{cases} \tag{1}$$

where $d_1 > 0$ and $d_2 > 0$ are the depths of the two wells and w is the wells' common width. The asymmetric Duffing potential is

$$V(x) = \frac{x^4}{4} - \frac{\lambda x^3}{3} - \frac{x^2}{2}. \tag{2}$$

This potential has two wells with relative minima at

$$x_{min1} = x_{min1}(\lambda) = \frac{\lambda}{2} - \sqrt{\frac{\lambda^2}{4} + 1}$$

2nd International Conference on Computation of Stochastic Mechanics, June 13-15, 1994, Athens, Greece, P.D. Spanos, ED. 1995

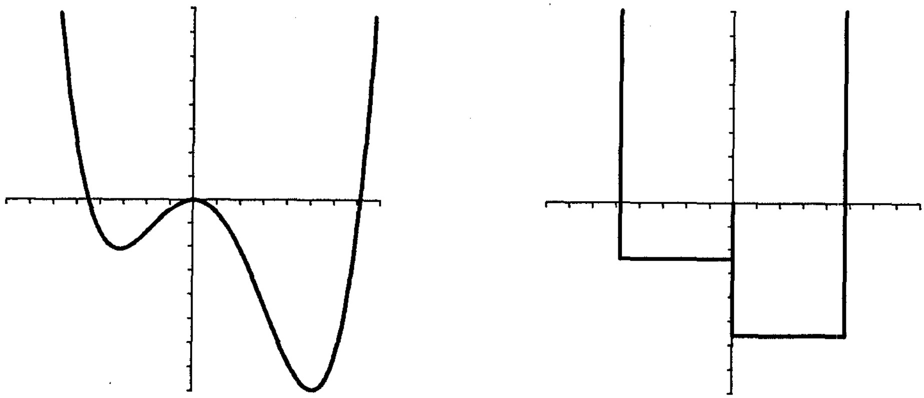

Figure 1. Examples of the asymmetric Duffing and double square well potentials.

and

$$x_{min2} = x_{min2}(\lambda) = \frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} + 1}$$

separated by a potential energy barrier with height $V(0) = 0$. The depth of the well at x_{min2} is $V(0) - V(x_{min2}) = -V(x_{min2})$. We have

$$\frac{d}{d\lambda}\{-V(x_{min2}(\lambda))\} = \frac{1}{3}\left(\frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} + 1}\right)^3$$

so the depth of the well at x_{min2} is a strictly increasing function of λ. Similarly, the depth of the well at x_{min1} is a strictly decreasing function of λ. Therefore, well 2 is deeper than well 1 for $\lambda > 0$, shallower for $\lambda < 0$ and of equal depth for $\lambda = 0$. Examples of the double square well potential and the asymmetric Duffing potential are shown in Figure 1.

The damped, noise-perturbed counterpart of $\ddot{x} = -V'(x)$ is the system defined by

$$\ddot{x} = -V'(x) + \gamma \dot{W}(t) - k\dot{x} \tag{3}$$

where $k > 0$, $\gamma > 0$, and $\dot{W}(t)$ is a Guassian white noise process with unit spectral density. Written as a set of first order stochastic differential equations, system (3) is

$$\begin{aligned} dv(t) &= -[V'(x) + kv(t)]\,dt + \gamma\,dW(t) \\ dx(t) &= v(t)\,dt \end{aligned} \tag{4}$$

where $W(t)$ is a Wiener process.

2. Probability Ratio Relative Stability

Relative stability can be defined as the probability ratio $\rho_{21} = p_2/p_1$ where p_1 is the probability of being in well 1, p_2 is the probability of being in well 2 and ρ_{21} is the relative stability of well 2 with respect to well 1 (van Kampen 1988). According to this definition, well 2 of a bistable system is stable relative to well 1 if the state of the system is more likely to be in well 2 than well 1. The probabilities p_1 and p_2 used in this approach are those of the stationary distribution of the system state. In general, these probabilities must be obtained by Monte Carlo simulation. However, for system (4), the density of the stationary distribution of the state (x, v) can be obtained as a solution of Kramers' equation (Gardiner 1990). This distribution is Boltzmann with density

$$p_s(x, v) = \mathcal{N} \exp\left(-\frac{2kV(x)}{\gamma^2} - \frac{kv^2}{\gamma^2}\right) \tag{5}$$

where $\mathcal{N}$ is a normalization constant. The density of the stationary distribution of the variable x alone is then

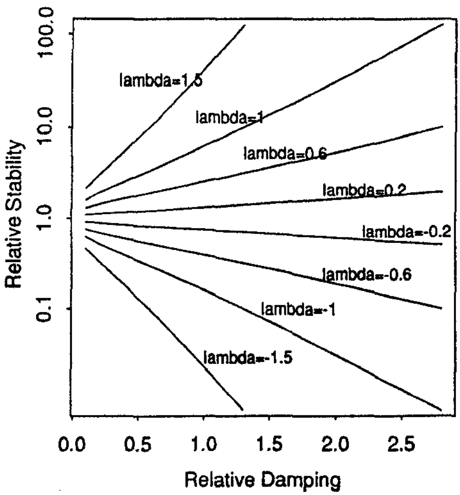

Figure 2. Probability ratio relative stability of the wells of the asymmetric Duffing potential for various values of asymmetry parameter λ and relative damping k/γ^2.

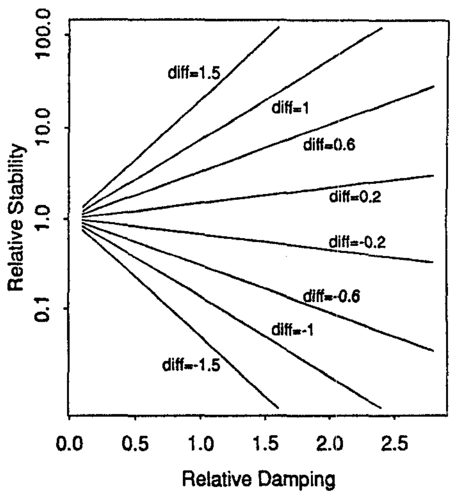

Figure 3. Probability ratio relative stability of the wells of the double square well potential for various values of relative depth diff = d_2 - d_1 and relative damping k/γ^2.

$$p_s(x) = \mathcal{N} \exp\left(-\frac{2kV(x)}{\gamma^2}\right) \tag{6}$$

where $\mathcal{N}$ is another normalization constant. For convenience it is assumed that the energy potential $V(x)$ is defined as in the two examples above such that one of its wells is set to the right of $x = 0$ while the other well is set to the left of $x = 0$. The left well is referred to as well 1 and the right well as well 2. It is also assumed that $V(0) = 0$ as in our two examples. The probability ratio relative stability ρ_{21} of well 2 with respect to well 1 is defined to be the stationary odds ratio

$$\rho_{21} = \lim_{t\to\infty} \frac{P\{x(t) > 0\}}{P\{x(t) < 0\}}$$

provided this limit exists. If the potential $V(x)$ increases sufficiently rapidly for $x \to \infty$ and $x \to -\infty$ as, for example, in the case of system (3) with either the double square well potential or the asymmetric Duffing potential, the limit does exist and

$$\rho_{21} = \frac{\int_0^\infty \exp(-2kV(x)/\gamma^2)\,dx}{\int_{-\infty}^0 \exp(-2kV(x)/\gamma^2)\,dx}. \tag{7}$$

The relative stability ρ_{21} of the wells of the asymmetric Duffing potential and of the double square well potential is plotted in Figures 2 and 3 for various values of k, γ and well parameters.

3. Flux Ratio Relative Stability

We now introduce a different measure ϱ_{21} of relative stability. First, express system (3) in reparameterized form:

$$\ddot{x} = -V'(x) + \varepsilon\gamma G(t) - \varepsilon k\dot{x}. \tag{8}$$

In system (8) we assume more generally that $G(t)$ is a colored Gaussian process with one-sided power spectrum $2\pi\Psi$ and unit variance. We assume that the potential energy $V(x)$ of the unperturbed ($\varepsilon = 0$) counterpart of system (8) has a hyperbolic saddle point at $(x, \dot{x}) = (0, 0)$ connected to itself by two homoclinic orbits, $z_1(t) = (x_1(t), \dot{x}_1(t))$ and $z_2(t) = (x_2(t), \dot{x}_2(t))$ (Wiggins 1992). In the case of the asymmetric Duffing potential these orbits are obtained directly from the Hamiltonian equation $\dot{x}_j^2(t)/2 + V(x_j(t)) = 0$ (Brunsden et al. 1989). The orbits of the double square well potential are found using this same equation in conjunction with a limiting process in which the potential $V(x)$ is approximated by a sequence of continuously differentiable potentials.

Given the system in (8), the relative stability ϱ_{21} of well 2 with respect to well 1 is defined to be

$$\varrho_{21} = \frac{\Phi_{1,s}}{\Phi_{2,s}} \tag{9}$$

where $\Phi_{1,s}$ and $\Phi_{2,s}$ are the standardized flux factors of the homoclinic orbits $z_1(t)$ and $z_2(t)$. For $0 \le \epsilon \ll 1$, $\epsilon\Phi_{j,s} + O(\epsilon^2)$ is proportional to the phase space transported across the pseudo-separatrix of well j, $j = 1,2$ (Wiggins 1992). Because phase space transport is the only means of escape from the well, phase space flux reflects the stability of the well and well 2 is more stable than well 1 when $\varrho_{21} > 1$. For $j = 1,2$ the (unstandardized) flux factor $\Phi_j \equiv \Phi_{j,s}A_j$ for well j of system (8) can be expressed (Frey & Simiu 1993a,b)

$$\Phi_j = E[(\gamma\sigma_j Z - kA_j)^+] \tag{10}$$

where Z is a standard Gaussian random variable,

$$\begin{aligned} A_j &= \int_{-\infty}^{\infty} \dot{x}_j^2(s)\,ds & (11)\\ &= 2\int_0^{x_{jo}} \dot{x}_j\,dx_j & (12)\\ &= 2(-1)^j \int_0^{x_{jo}} \sqrt{-2V(x)}\,dx & (13) \end{aligned}$$

with $V(x_{jo}) = 0$ and

$$\sigma_j^2 = \int_0^{\infty} S_j^2(\nu)\Psi(d\nu). \tag{14}$$

Expression (12) for A_j can be interpreted geometrically as the area enclosed by the homoclinic orbit $z_j(t) = (x_j(t), \dot{x}_j(t))$. Thus A_j in (10) is a geometrical measure of the depth and breadth of well j. $S_j(\nu)$ in (14) is the filter function of orbit $z_j(t)$; that is, $S_j(\nu) = |H_j(\nu)|$ where $H_j(\nu)$ is the Fourier transform

$$H_j(\nu) = \int_{-\infty}^{\infty} h_j(t)e^{-i\nu t}\,dt$$

of the filter impulse response $h_j(t) = \dot{x}_j(-t)$.

Proposition 1: If $G(t)$ in (8) is white Gaussian noise then $\sigma_j^2 = A_j$ for $j = 1,2$ where A_j is the area enclosed by the homoclinic orbit of well j.

Proof: If $G(t)$ is a white Gaussian noise process then its spectrum is uniform, $\Psi(d\nu) = d\nu$, and, using Plancherel's formula,

$$\begin{aligned} \sigma_j^2 &= \frac{1}{\pi}\int_0^{\infty} S_j^2(\nu)\,d\nu \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H_j(\nu)|^2\,d\nu \\ &= \int_{-\infty}^{\infty} h_j^2(t)\,dt \\ &= \int_{-\infty}^{\infty} \dot{x}_j^2(t)\,dt \end{aligned}$$

where the last equality is obtained using $h_j(t) = \dot{x}_j(-t)$. The result then follows from (11). □

Using the fact that Z in (10) is a standard Gaussian random variable, we have

$$\Phi_j = \gamma\sigma_j(\phi(\kappa_j) - \kappa_j \mathrm{erfc}(\kappa_j))$$

and

$$\Phi_{j,s} = k\frac{\phi(\kappa_j) - \kappa_j \mathrm{erfc}(\kappa_j)}{\kappa_j} \tag{15}$$

where $\kappa_j = (k/\gamma)/(A_j/\sigma_j)$, $\phi(z)$ is the standard Gaussian density and $\mathrm{erfc}(z)$ is the complementary error function,

$$\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$$

and

$$\mathrm{erfc}(z) = \int_z^{\infty} \phi(x)\,dx.$$

Proposition 2: For system (8) with colored Gaussian noise $G(t)$ the standardized flux $\Phi_{j,s}$ is a decreasing function of A_j.

Proof: The derivative of the standardized flux $\Phi_{j,s}$ with respect to κ_j is

$$\frac{\partial\Phi_{j,s}}{\partial\kappa_j} = -k\frac{\phi(\kappa_j)}{\kappa_j^2} < 0.$$

κ_j is proportional to the area A_j enclosed by the well's homoclinic orbit so $\Phi_{j,s}$ is a decreasing function of A_j. □

The extent of the homoclinic orbit of a well is determined by the size of the well. The broader the well, the greater the extent of the homoclinic orbit in the direction of the phase space variable x. The deeper the well, the greater the maximum velocity on the homoclinic orbit and the greater the extent of the orbit in the direction of the phase space variable v. Thus the area A_j encompassed by the homoclinic orbit reflects both the depth and breadth of a potential well; if either the depth or the breadth of the well is increased then A_j increases. Therefore, Proposition 2 establishes that the flux ratio definition of ϱ_{21} is order consistent; broader, deeper wells are more stable than narrower, shallower wells.

Proposition 3: The relative stability ϱ_{21} of well 2 with respect to well 1 for the perturbed Newtonian system (8) with colored Gaussian forcing $G(t)$ is

$$\varrho_{21} = \frac{\phi(\kappa_1)/\kappa_1 - \mathrm{erfc}(\kappa_1)}{\phi(\kappa_2)/\kappa_2 - \mathrm{erfc}(\kappa_2)} \tag{16}$$

where $\kappa_j = (k/\gamma)(A_j/\sigma_j)$ for $j = 1,2$.

Proof: Insert expression (15) for the standardized flux factor in (9). □

Limiting expressions are available for ϱ_{21}. If the damping, k, is small relative to the apparent noise strength $\gamma\sigma_j$ in each well, then $\kappa_j \ll 1$, $j = 1,2$ and

$$\varrho_{21} \approx \frac{\kappa_2}{\kappa_1} = \frac{A_2}{A_1}\frac{\sigma_1}{\sigma_2}. \tag{17}$$

If z is large then

$$\mathrm{erfc}(z) \approx \frac{\phi(z)}{z}\left(1 - \frac{1}{z^2} + \frac{1\cdot 3}{z^4} - \frac{1\cdot 3\cdot 5}{z^6} + \cdots\right)$$

where the error is less than the last term used. Thus, if the damping is large relative to the apparent noise strength in each well, then $\kappa_j \gg 1$, $j = 1,2$ and

$$\varrho_{21} \approx \frac{\phi(\kappa_1)/\kappa_1^3}{\phi(\kappa_2)/\kappa_2^3}. \tag{18}$$

The relative stability ϱ_{21} is given in terms of κ_j, $j = 1,2$ in all three expressions (16), (17) and (18). For the case of white Gaussian noise, $\kappa_j = (k/\gamma)(A_j/\sigma_j)$ can be simplified using Proposition 1. Since $\sigma_j = \sqrt{A_j}$,

$$\kappa_j = \frac{k}{\gamma}\sqrt{A_j}. \tag{19}$$

Expressions (16), (17) and (18) for ϱ_{21} are applicable for colored Gaussian forcing. Using expressions similar to (10), the flux ratio relative stability ϱ_{21} can accommodate processes $G(t)$ which are shot noises and, more generally, any filtered independent increment process (Frey & Simiu 1993b). Useful expressions for ϱ_{21} can even be obtained when $G(t)$ is a deterministic function (Frey & Simiu 1993a).

Asymmetric Duffing potential: The area A_j can be expressed in closed form for the wells of the asymmetric Duffing potential (2) using expression (13) for A_j. The result is

$$\begin{aligned} A_j &= \frac{4}{3}\sqrt{2}\lambda A^2\left(\sin^{-1}\frac{\lambda}{3A} + (-1)^j\frac{\pi}{2}\right) \\ &\quad + \frac{4}{3} + \frac{4\lambda^2}{9} \end{aligned} \tag{20}$$

where $A^2 = \lambda^2/9 + 1/2$ (Brunsden et al. 1989). Evidently, $A_2 > A_1$ for $\lambda > 0$. The homoclinic orbits $x_1(t)$ and $x_2(t)$ for the asymmetric Duffing potential can be obtained from the Hamiltonian equation $V(x) + \dot{x}^2/2 = 0$. We have

$$x_1(t) = \frac{1}{A\cosh t + \lambda/3}$$

and

$$x_2(t) = \frac{1}{A\cosh t - \lambda/3}.$$

The corresponding velocity components of the orbits are

$$\dot{x}_1(t) = \frac{A\sinh t}{(A\cosh t + \lambda/3)^2}$$

and

$$\dot{x}_2(t) = \frac{A\sinh t}{(A\cosh t - \lambda/3)^2}.$$

The velocity components are needed for the calculation of σ_1 and σ_2 in (14) in the case of colored Gaussian noise. For white Gaussian noise, these calculations can be avoided using Proposition 1. For purpose of comparison with ρ_{21} we assume $G(t)$ is white Gaussian noise and calculate ϱ_{21} using (19) with (20) in (16). The results are shown in Figure 4.

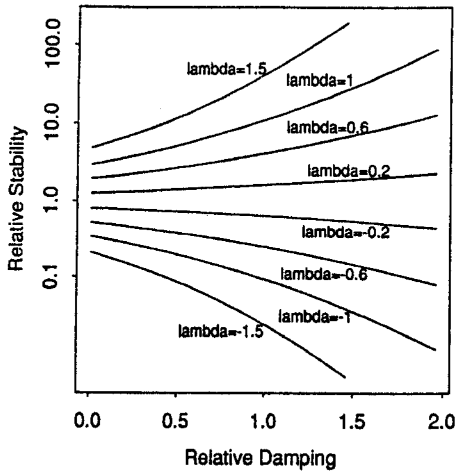

Figure 4. Flux ratio relative stability of the wells of the asymmetric Duffing potential for various values of asymmetry parameter λ and relative damping k/γ.

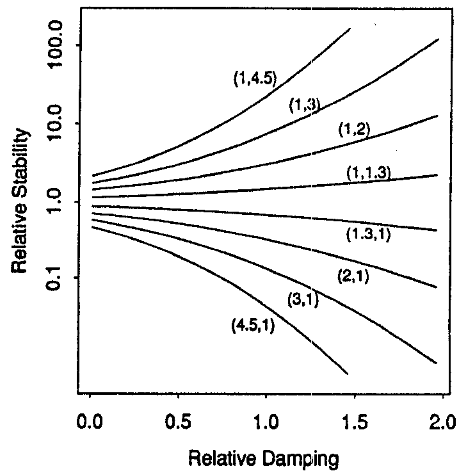

Figure 5. Flux ratio relative stability of the wells of the double square well potential for various well sizes ($wd_1^{1/2}, wd_2^{1/2}$) and relative damping k/γ.

Double square well potential: The speed is constant on the homoclinic orbits of the wells of this potential. Hence the homoclinic orbits are rectangles and the area A_j is the width w of the well multiplied by twice the speed along the homoclinic orbit. This speed can be found from the Hamiltonian equation $\dot{x}_j^2(t)/2 + V(x_j(t)) = 0$. $V(x_j(t)) = d_j$ so $A_j = 2w\sqrt{2d_j}$. Again we assume $G(t)$ is a white Gaussian noise process and calculate ϱ_{21} using (19) with $A_j = 2w\sqrt{2d_j}$ in (16). The results are shown in Figure 5.

Comparisons of Figures 2 and 4 and Figures 3 and 5 show that ρ_{21} and ϱ_{21} perform similarly. However they measure essentially different features of the potential wells of a dynamical system and neither measure can be used to numerically approximate the other. Note, for example, that for the probability ratio relative stability ρ_{21}, the natural relative damping is k/γ^2 since this is the factor that appears in expression (7) for ρ_{21}. By contrast, the flux ratio relative stability is expressed in terms of k/γ in, for example, Proposition 3 for ϱ_{21}. Thus for ϱ_{21} the natural expression of relative damping is k/γ rather than k/γ^2. As a second example of the differences between ϱ_{21} and ρ_{21}, consider the double square well system. Figure 3 shows that when the well widths are equal, only the difference $d_2 - d_1$ in well depths enters into the calculation of ρ_{21}. Figure 5 shows that for the same case, ϱ_{21} depends on each of the well depths d_1 and d_2 as well as the common well width w via the ordered pair $(w\sqrt{d_1}, w\sqrt{d_2})$.

4. Blowtorch Theorem

The blowtorch theorem concerns the probability ratio relative stability ρ_{21} of the potential wells of a bistable system (van Kampen 1988). The theorem states that if in some interval I belonging to well 2 the effective temperature is increased above that outside the interval I (as if by a blowtorch), then the relative stability of well 2 is decreased with respect to well 1. The temperature referred to by the theorem is that of the "heat bath" which is the source of the diffusion of the system state. In the case of system (3) the role of temperature is played by the noise strength γ. Flux ratio relative stability also satisfies a blowtorch theorem. A restricted form of the theorem is presented here for clarity.

Theorem 1: Let system (8) be bistable with potential energy $V(x)$ and suppose the forcing process $G(t)$ is white Gaussian noise. Assume that well 2 of $V(x)$ is contained within the interval $(0, \infty)$ and that well 1 is contained within $(-\infty, 0)$. Let I be a subset of $(0, \infty)$ and let the noise strength γ be state-dependent, $\gamma = \gamma(x) = \gamma_0 + \gamma_1 1_I(x)$, where $\gamma_0 > 0$, $\gamma_1 \geq 0$ and $1_I(x) = 1$ if $x \in I$ and $I(x) = 0$ otherwise. Then the flux ratio relative stability ϱ_{21} of well 2 with respect to well 1 is a nonincreasing function of γ_1 for $\gamma_1 \geq 0$.

Theorem 1 is stated and proved under more general conditions in a forthcoming report.

5. Discussion and Conclusions

We have introduced a new definition of relative stability for potential wells. This definition is attractive for several reasons. First, it has reciprocal symmetry ($\varrho_{21} = 1/\varrho_{12}$) and it meets the essential test of consistent order established in Proposition 2; broader, deeper wells are more stable. Both of these properties are evident in Figures 4 and 5. Second, it exhibits the desirable blowtorch property stated in Theorem 1. Third, it has a clear geometrical foundation in terms of phase space transport. Fourth, it admits closed-form expressions for very general forms of damping and external forcing—both stochastic and deterministic. This holds true for additive forcing and more generally for multiplicative forcing. We believe no other definition of relative stability is so convenient. Closed-form expressions are available for the probability ratio relative stability ρ_{21} in only the simplest of cases. For deterministic forcing ρ_{21} is not even meaningful since no probability is involved. Thus, for instance, flux ratio relative stability can be used to compare the effects of random and nonrandom forcing but probability ratio relative stability cannot. The phase space transport definition of relative stability has the following limitations. The concept of phase space transport across a pseudo-separatrix is not generally applicable in multi-dimensional dynamical systems. Also, the definition of ϱ_{21} is founded on asymptotic ($\varepsilon \to 0$) expressions for the phase space flux. Fortunately this is an important case in engineering applications. Finally, while ρ_{21} and ϱ_{21} share important properties, they measure different forms of relative stability; knowledge of ϱ provides no information about the numerical magnitude of ρ and conversely knowledge of ρ provides no information about ϱ. Both ρ_{21} and ϱ_{21} are time-invariant measures of relative stability, as are all the measures mentioned in the our introduction. Time-invariant measures of stability are best suited to system design and analysis. Time-dependent measures of stability, while not addressed here, might serve as the basis for dynamical system control. Further exploration of measures of relative stability based on phase space flux and the related concept of the generalized Melnikov function is underway.

Acknowledgement

This work was supported by the Ocean Engineering Division of the U.S. Office of Naval Research under Grants N00014-94-0028 and N00014-94-0284.

References

Ariaratnam, S.T. & Xie, W.-C., "Lyapunov Exponents and Stochastic Stability of Two-Dimensional Parametrically Excited Random Systems," *J. Appl. Mech.*, **60** (1993) 677-682.

Brunsden, V., Cortell, J. & Holmes, P.J., 1989, "Power Spectra of Chaotic Vibrations of a Buckled Beam," *Jour. of Sound and Vibration*, **130**(1) 1-25.

Frey, M.R. & Simiu, E., 1993, "Noise-induced chaos and phase space flux," *Physica D*, **63** 321-340.

Frey, M.R. & Simiu, E., 1993, "Deterministic and stochastic chaos," in *Computational Stochastic Mechanics*, A.H-D. Cheng and C.Y. Yang (eds.), Computational Mechanics Publications, Ashurst, United Kingdom.

Gardiner, C.W., 1990, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, Second Edition, Springer Verlag, Inc., New York.

Howard, J.E., 1990, "Spectral Stability of Relative Equilibria," *Celestial Mechanics and Dynamical Astronomy*, 48 267-288.

Maddocks, J.H., 1991, "On the Stability of Relative Equilibria," *IMA Jour. of Appl. Math.*, 46 71-99.

van Kampen, N.G., 1988, "Relative Stability in Nonuniform Temperature," *IBM Jour. Res. Develop.*, 32(1) 107-111.

Wiggins, S., 1992, *Chaotic Transport in Dynamical Systems*, Springer, Inc., New York.