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Exits in second-order nonlinear systems driven by dichotomous noise

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ABSTRACT: We consider a wide class of lightly damped second-order differential equations with double-well potential and small coin-toss square wave dichotomous noise. The behavior of these systems is similar to that of their harmonically or quasiperiodically driven counterparts: depending upon the system parameters the steady-state motion is confined to one well for all time or experiences exits from the wells. This similarity suggests the application to the stochastic systems of a Melnikov-based approach originally developed for deterministic systems. This approach accommodates both additive and multiplicative noise. It yields a generalized Melnikov function which is used to obtain (i) a very useful simple condition guaranteeing the non-occurrence of exits from a well, and (ii) very weak lower bounds for the mean time of exit from a well and for the probability that exits will not occur during a specified time interval.

1. INTRODUCTION

Numerous studies have been devoted, especially in the last decade, to dynamical systems driven by dichotomous noise, which is characterized primarily by whether it is "on" or "off," or whether it is "up" or "down" (Cohen 1962; Kitahara et al. 1980; Sancho 1984; Behn and Schiele 1989; Janeczko and Wajnryb 1989; L'Heureux, Kapral and Bar-Eli 1989; Irwin, Fraser and Kapral 1990; Kapral and Fraser 1993; Porrà, Masoliver and Lindenberg 1993; L'Heureux 1993). One example are systems where the excitation exceeds or does not exceed a specified threshold — situations described as "on" and "off," respectively. To our knowledge, analytical procedures applicable to systems driven by dichotomous noise are available only for dynamical systems that are linear or of first order, or that can be reduced to a linear or first-order system.

In this note we consider a class of nonlinear, second-order differential equations perturbed by a damping term and dichotomous noise. We present a Melnikov-based procedure which, given a set of system parameters, can establish whether exits from a potential well are possible. If exits can occur, the procedure can be used to obtain lower bounds for the mean exit time and the probability that exits will not occur during a specified time interval.

The Duffing equation belongs to our class of systems, and is considered here for specificity. We assume that the dichotomous noise is of the coin-toss square wave type (Cohen 1963, Porrà et al. 1993).

The noise may be represented, to any desired approximation, by a stochastic process consisting of the sum of N harmonic terms with random parameters, where N is finite, albeit large. Examples of similar representations of various types of noise are available in Shinozuka (1971) and Frey and Simiu (1993a). This representation need not be carried out explicitly, but can be invoked to show that the system driven by the approximating stochastic process may be suspended in an extended phase space of dimension N+2, in which it is autonomous (Beigie, Leonard and Wiggins 1991, Wiggins 1992, Frey and Simiu 1993b). Under certain conditions, the saddle point of the integrable system persists under perturbation in a slice through the extended phase space. However, the stable and unstable manifolds emanating from the persisting saddle point no longer coincide, as they do on the homoclinic orbits of the unperturbed system. The distance between the stable and unstable manifolds of the perturbed system is proportional, to first order, to the generalized Melnikov function (GMF). By virtue of the Smale-Birkhoff

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theorem, the necessary condition for the occurrence of chaos (i.e., the necessary condition for the largest Lyapounov exponent to be positive or, equivalently, for the system to be sensitive to initial conditions) is that the GMF have simple zeros. In that case the stable and unstable manifolds intersect an infinite number of times and form lobes, by which chaotic transport between wells is effected. No chaotic transport into a well can occur unless the GMF has simple zeros (Wiggins 1992; Frey and Simiu 1993b). Moreover, for relatively high damping-to-forcing ratios, the time needed for a particle to cross a pseudoseparatrix is on the average equal, to within a factor of order one, to the time between successive zeros with positive slope of the GMF. This observation allows the estimation of a weak lower bound to the mean time of exit from a well (Simiu and Frey 1994). To assess the weakness of the lower bound an analytical expression is derived for the relation between a similar lower bound and the mean exit time for the case of excitation by white noise.

Section 2 describes the class of systems to which our approach is applicable, and the noise process. Section 3 describes the generalized Melnikov function (GMF) induced by the noise process. It discusses (i) a GMF-based criterion guaranteeing the non-occurrence of exits, and (ii) lower bounds for the mean exit time and the probability of no exits during a specified time. Section 4 includes results of numerical simulations for non-chaotic and chaotic stochastic motions, which further illustrate the usefulness of the necessary condition for the occurrence of chaos. Section 5 presents our conclusions.

2. DYNAMICAL SYSTEMS AND NOISE DESCRIPTION

2.1 Dynamical systems

The dynamical systems are described by the equation

$$\ddot{x} = -V'(x) + \epsilon[\gamma G(t) - \beta\dot{x}] \tag{1}$$

where $\epsilon \ll 1$ and $V(x)$ is a potential function. The assumptions concerning the unperturbed system ($\epsilon = 0$) are: (i) the unperturbed equations are integrable; (ii) the potential $V(x)$ has the shape of a double well, and the unperturbed system has three fixed points: two centers (one at the bottom of each well), and a saddle point at the top of the barrier between the two wells; and (iii) the saddle point is connected to itself by homoclinic orbits.

For specificity we consider the case of the Duffing equation with potential

$$V(x) = x^4/4 - x^2/2 \tag{2}$$

2.2 Noise description

The expression for the dichotomous coin-toss square-wave noise is

$$G(t) = a_n \qquad [\alpha+(n-1)]t_o < t \le (\alpha+n)t_o, \tag{3}$$

where $n = \ldots,-2,-1,0,1,2,\ldots$ is the set of integers, α is a random variable uniformly distributed between 0 and 1, a_n are independent random variables that take on the values -1 and 1 with probabilities 1/2 and 1/2, respectively, and t_o is a parameter of the process $G(t)$.

Note that the process $G(t)$ may be represented in terms of Heaviside functions. Therefore, $G(t)$ can be approximated arbitrarily closely by substituting for the Heaviside functions appropriate well-behaved functions, e.g., functions of the type

$$H_m = 1/2 + (1/\pi)\tan^{-1}(mx) \tag{4}$$

where m is sufficiently large (Kanwal 1983). Alternatively, normal cumulative distribution functions with sufficiently small standard deviations may be used.

3. GENERALIZED MELNIKOV FUNCTION (GMF)

The GMF may be used to obtain a simple criterion guaranteeing that exits from a well cannot occur. It can also be used to estimate a transport-time index. The index is a lower bound for the mean exit time, and can be used to estimate a lower bound for the probability that exits from a well will not occur within a specified time interval.

3.1 Expression for the Melnikov function

The GMF is defined by the expression

$$M(t) = -\beta\int_{-\infty}^{\infty}\dot{x}_s^2(\tau)d\tau + \gamma\int_{-\infty}^{\infty}h(\tau)G(t-\tau)d\tau \tag{5}$$

where $\dot{x}_s$ is the ordinate in the $x,\dot{x}$ phase plane of the unperturbed system's homoclinic orbit, and the filter in the convolution integral of Eq. 5 is $h(t) = \dot{x}_s(-t)$ (Frey and Simiu 1993b). The theorem that proves persistence under small

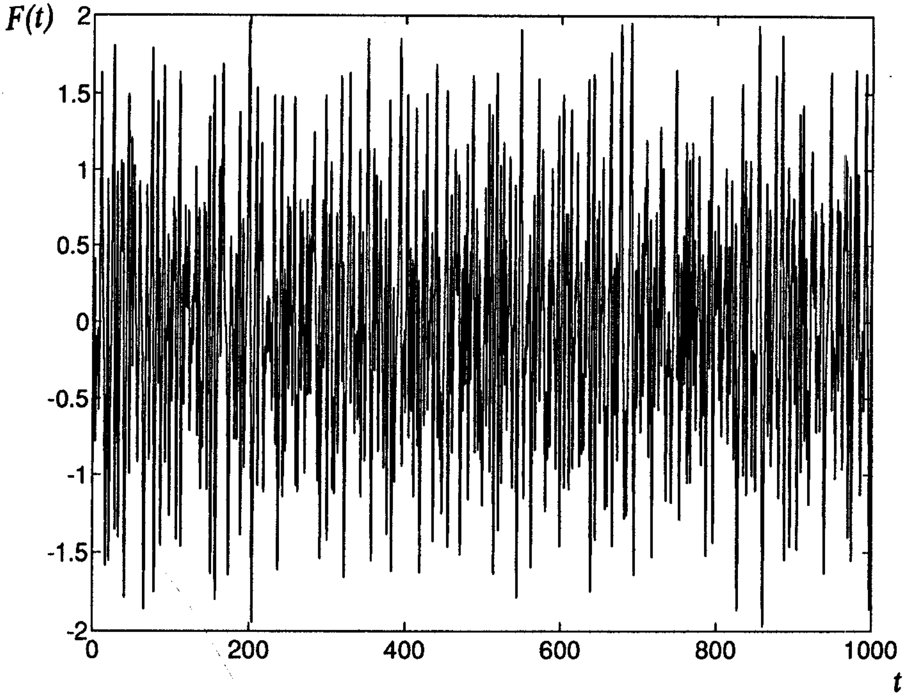

Fig. 1. Function F(t) for $t_o = 1$.

perturbation of the unperturbed system's saddle point requires G(t) to be sufficiently smooth. However, in practice this requirement may be relaxed. To show this G(t) may be approximated in terms of well-behaved functions, e.g., the functions H_m (Eq. 3), where m is sufficiently large for the errors in the representation of G(t) to be negligible. A similar approach was demonstrated in some detail in (Frey and Simiu 1993b).

Equation 5 is valid for the case of additive noise. If in Eq. 1 multiplicative noise $f(x,\dot{x})G(t)$ is considered instead of the additive noise $\gamma G(t)$, then M(t) is obtained simply by replacing in Eq. 5 the filter $h(t)=\dot{x}_s(-t)$ by the filter

$$h_1(t) = \dot{x}_s(-t)f(x_s(-t),\dot{x}_s(-t)) \quad (6)$$

(Simiu & Frey 1994).

For the Duffing oscillator (Eqs. 1 and 2)

$$\dot{x}_s(t)=(2)^{1/2}\mathrm{sech}(t)\tanh(t) \quad (7)$$

and the GMF is

$$M(t)=-4\beta/3 + (2)^{1/2}\gamma F(t) \quad (8)$$

$$F(t)\approx\sum_{n=-\ell}^{\ell} a_n\{-\mathrm{sech}[(n+\alpha)t_o-t]+\mathrm{sech}[(n+\alpha-1)t_o-t]\} \quad (9)$$

where ℓ is sufficiently large for the error due to the assumption that ℓ is finite to be negligibly small.

A realization of the random process F(t) is represented in Fig. 1 for $t_o=1$. For this case the standard deviation of F(t), obtained from Eqs. 8 and 9, is $\sigma_F=0.772$. For $t_o=3.14$, $\sigma_F=0.962$. Note that

$$M(t)/[(2)^{1/2}\gamma] = F(t) - 4\beta/[(2)^{1/2}\gamma]. \quad (10)$$

We refer to the left-hand-side of this equation as the rescaled GMF.

3.2 Criterion guaranteeing non-occurrence of exits

The area under the curve $x_s(t)$ (Eq. 7) in a half-plane is $(2)^{1/2}$. It then follows from the definition of F(t) (Eqs. 5 and 8) that $-2<F(t)<2$. (The probabilities of occurrence of noise realizations for which $|F(t)|=2$

are zero, but they are non-zero for $|F(t)|=2-\delta$, $\delta \ll 1$.) By the Smale-Birkhoff theorem, the necessary condition for chaos is that M(t) have simple zeros. If

$$\beta/\gamma > 3/(2)^{1/2} = 2.121, \tag{11}$$

then this condition cannot be satisfied, and chaotic transport from one well to the other cannot occur, no matter how long the waiting time. Equation 11 is a stability criterion applied to a fairly complex stochastic nonlinear differential equation. Its simplicity, in our view, is remarkable. We recall that this criterion was obtained by applying to a stochastic equation a result of chaotic dynamics theory.

3.3 Lower bound for mean exit time

For sufficiently small ϵ the intersection with a phase space slice of the stable and unstable manifolds exhibits lobes whose ordinates are, to first order, proportional to the GMF (Wiggins 1992). A line of constant ordinate $4\beta/[3(2)^{1/2}\gamma]=0.9428\ \beta/\gamma$ in Fig. 1 is the zero line for the rescaled GMF. The counterparts in Fig. 1 of the entraining lobes (lobes that will cross or have crossed into the interior of the pseudoseparatrix) are the small areas between the zero line of the rescaled GMF and the positive part of the rescaled GMF. The counterparts of the detraining lobes (lobes that will cross or have crossed into the exterior of the pseudoseparatrix) are the relatively large areas between the zero line of the rescaled GMF and the negative part of the rescaled GMF. (For details on entraining and detraining lobes see Beigie, Leonard and Wiggins 1991). For sufficiently high ratios β/γ the zero upcrossings of the function M(t) are rare events. We denote the mean time between those upcrossings by τ_{Mu} and view it as a transport-time index. On average, the time of transport across the pseudoseparatrix is, to within a factor of order one, equal to τ_{Mu}. τ_{Mu} is smaller than and is therefore a lower bound for the mean exit time, τ_{ex}, corresponding to an initial position at or near the bottom of a well.

It is clear from its definition that the transport-time index is a weak lower bound for τ_{ex}. To illustrate this, consider the case where in Eq. 1 G(t) denotes white noise with autocorrelation equal to the Dirac delta function, and $\epsilon\gamma$ denotes the noise intensity. The mean time between potential barrier crossings can be shown to be

$$\tau_{ex,wn}=-(4\pi\beta/\epsilon)^{1/2}(2/\gamma)\int_{-o}^{\infty}\exp[-2\beta/\epsilon/\gamma^2 V(x)]dx \tag{12}$$

We assume $\epsilon=0.1$, $\beta=0.1$, $\gamma=0.025$. For these values $\tau_{ex,wn}\approx 10^{350}$.

For white noise excitation the GMF can be defined by considering excitation by a uniform broadband power spectrum from $\omega=0$ to $\omega=\omega_f$. In the limit of small ϵ and large ω_f the standard deviation of the GMF is $\gamma\sigma_Z$, where

$$\sigma_Z{}^2=\int_0^{\infty}S^2(\omega)d\omega \tag{13}$$

and $S(\omega)$ is the Fourier transform of h(t) (Frey and Simiu 1993b). The ratio of the mean to the standard deviation of the GMF is then $k=\beta I/(\gamma\sigma_Z)$, where I is the value of the first integral in the right-hand side of Eq.5. (For the Duffing equation $S(\omega)=(2)^{1/2}\pi\omega\,\mathrm{sech}(\pi\omega/2)$, $\sigma_Z=2(\pi/3)^{1/2}=2.047$, and $I=4/3$.) The GMF is a Gaussian process, and the mean upcrossing rate of the threshold k is

$$\tau_{Mu,wn} = \alpha_1\exp(c_2\beta^2/\gamma^2) \tag{14}$$

$$1/\alpha_1=(1/2\pi)[\int_0^{\infty}\omega^2S^2(\omega)d\omega]/[\int_0^{\infty}S^2(\omega)d\omega] \tag{15}$$

$c_2=I^2/(2\sigma_Z{}^2)$ (Rice 1954). For our parameters $\tau_{Mu,wn}=160$, a very weak lower bound indeed.

3.4 Upper bound to probability that exits occur during specified time interval

We assume again that the ratio β/γ is sufficiently high that upcrossings of the threshold 0.9428 β/γ by the function F(t) are rare events. The probability that no upcrossing occurs during a specified time interval T can be written as

$$p_T\approx\exp(-T/\tau_{Mu}) \tag{16}$$

Since $\tau_{Mu}<\tau_{ex}$, p_T is an approximate lower bound to the probability that exits from a well will not occur during the time interval T. For example, let $t_o=1$ and $\beta/\gamma=1.9$. From Fig. 1 $\tau_{Mu}\approx165$. For T=20, Eq. 17 then yields $p_T\approx0.89$.

4. NUMERICAL SIMULATIONS

Figures 2a and 2b show time histories of the motion for the Duffing equation excited

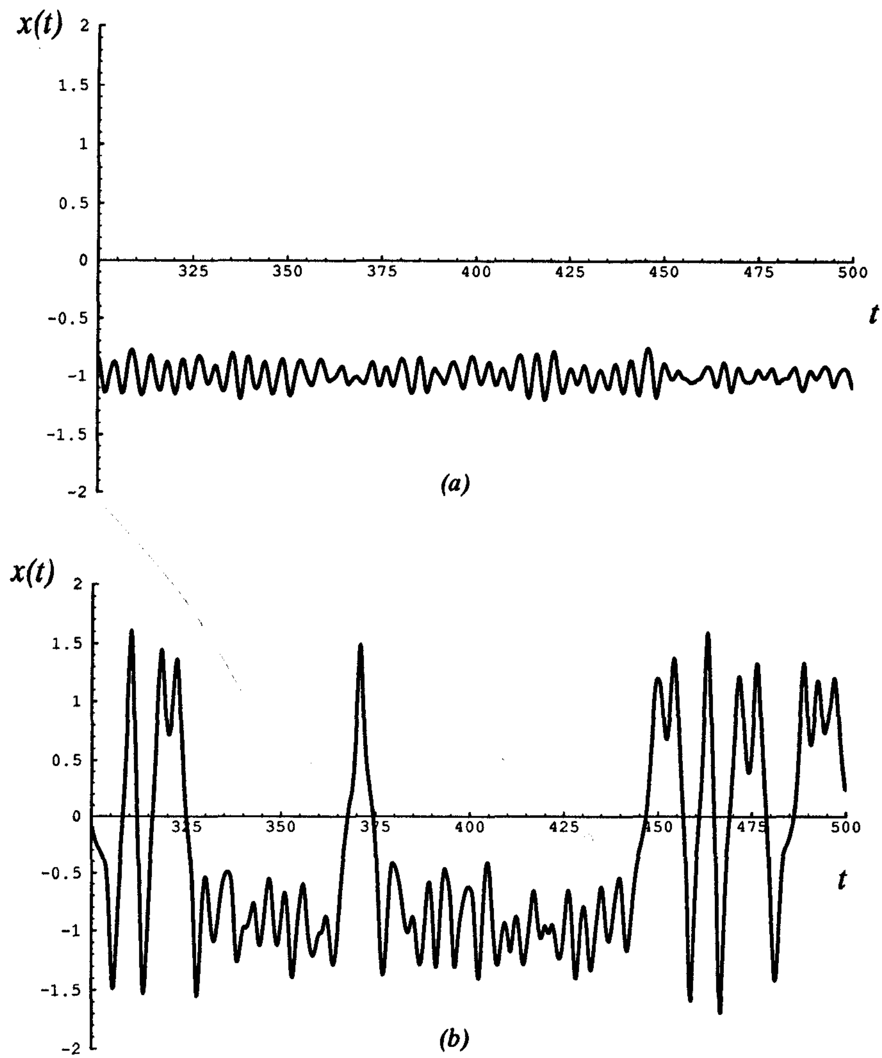

Fig. 2. Realizations of stochastic motions induced by dichotomous noise: (a) non-chaotic motion; (b) chaotic motion.

by realizations of the dichotomous noise G(t) with $t_0=1$ (Eq. 3), and corresponding to the parameters $\epsilon=1$, $\beta=0.15$, and $\beta/\gamma=2.13>2.121$ (see Eq. 11), and $\beta/\gamma=0.625<2.121$. The motion in Fig. 2a is confined to one well, as predicted by Eq. 11, and differs from its counterpart in the harmonically excited Duffing oscillator by being irregular, a result of the randomness of the excitation. The chaotic motion of Fig. 2b is strikingly similar to chaotic motions induced by harmonic or quasiperiodic excitation. Underlying the commonality of the stochastic and deterministic systems is the existence in both cases of stable and unstable manifolds whose behavior, assessed by the Melnikov distance, controls the system behavior.

Note that, as for the Duffing equation with harmonic forcing (Moon, 1987), the necessary condition for the occurrence of chaos is helpful in the search for chaotic regions even for relatively large ϵ.

Sensitivity to initial conditions (i.e., the positivity of the largest Lyapounov exponent) was verified numerically for the motion of Fig. 2b.

5. CONCLUSIONS

We showed that, for a class of second-order bistable differential equations, forcing by dichotomous noise induces behavior that has striking similarities with behavior induced by harmonic or quasiperiodic forcing. For certain regions of parameter space, both the stochastic system driven by noise and the deterministic system driven harmonically experience behavior that may be chaotic or non-chaotic. Non-chaotic behavior precludes the occurrence of exits from the potential wells. However, if the behavior is chaotic, exits from the wells become possible via the mechanism of chaotic transport by phase space slice lobes. A necessary condition for the occurrence of chaos in the deterministic and stochastic systems is the existence of simple zeros in, respectively, the Melnikov function (which is a deterministic function) and the GMF (which is a stochastic process). This parallelism suggested extending to our stochastic differential equations an approach based on the theory of chaotic dynamics that was originally developed for deterministic systems. This approach accommodates both additive and multiplicative noise, and yields a remarkably simple criterion guaranteeing the non-occurrence of exits. We defined a transport-time index, which is a weak lower bound to the mean exit time from a well, and obtained a weak lower bound to the probability of non-occurrence of exits during a specified time interval. We showed that the bounds we obtained are very weak. In spite of that weakness the lower bounds may be useful in some applications, particularly for the relative assessment of the effect on chaotic transport of various features of the noise.

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