Chaotic Transitions in Deterministic and Stochastic Dynamical Systems: Applications of the Melnikov Method in Engineering, Physics, and Neuroscience

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ABSTRACT: The classical Melnikov method provides information on the behavior of deterministic planar systems that may exhibit transitions, i.e., escapes from and captures into preferred regions of phase space. This paper describes and illustrates a unified treatment of deterministic and stochastic systems that extends the applicability of the classical Melnikov method to physically realizable stochastic planar systems with additive, state-dependent, colored, or dichotomous noise. The extended method yields the novel result that motions with transitions are chaotic for either deterministic or stochastic excitation, explains the role in the occurrence of transitions of the system and excitation characteristics, and is a powerful modeling and identification tool.

INTRODUCTION

The Melnikov method is a unified framework for the study of transitions and chaos in a wide class of deterministic and stochastic nonlinear planar dynamical systems with restoring force derived from a multi-well potential. Its applications span a broad spectrum of problems in engineering and the applied sciences. In this paper we review fundamental results of Melnikov theory and, to give the reader a sense of its capabilities, present a number of typical applications.

As an example we consider the system

$$\ddot{x} = -V'(x) + \varepsilon(G(t) - \beta x)$$

(1)

where \(\varepsilon\) is small, \(G(t)\) is a sufficiently well behaved forcing function, \(\beta > 0\), and \(V(x)\) is a double-well potential (Fig. 1a).

We may use the notations \(x = x_1, \dot{x} = x_2\). The unperturbed counterpart of Eq. 1 corresponds to the case \(\varepsilon = 0\). For this case two types of generic motion can occur: (1) motions that evolve around \(C\) and \(C'\) and never cross the potential barrier; and (2) motions that evolve around the saddle point \(O\) and cross that barrier periodically (Fig. 1b). The homoclinic orbits, which approach \(O\) as \(t \to \pm \infty\), are non-generic and separate the two types of motion. No motion is possible from outside the core defined by the separatrix into the core, and vice-versa.

We now assume that in the perturbed (i.e., forced, dissipative system) (Eq. 1), the forcing is harmonic.

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For intermediate excitation amplitudes, and for restricted sets of initial conditions and excitation frequencies, the steady-state motion is irregular, even though the system is fully deterministic; hence the term deterministic chaos. The motion evolves about one of the centers, then it undergoes successive transitions, i.e., it changes successively to motion about another center (Fig. 2c). Transitions in such irregular, deterministic motion are referred to as chaotic. A transition away from motion in a potential well is called an escape. For a large number of systems it is required that steady-state motions occur within a region called safe, that is, that they not cross a potential barrier. For planar systems subjected to periodic excitation an analytical condition that guarantees the non-occurrence of transitions involves a function - the Melnikov function - consisting of a sum of two terms related, through the system’s potential, to the dissipation and excitation terms.

Melnikov’s method was extended to quasiperiodically excited systems (Wiggins, 1988) and stochastic systems (Frey and Simiu, 1993). The stochastic counterpart of the Melnikov function is a Melnikov process. Through the modulus of the Melnikov transfer function -- a function of frequency that depends upon the system’s potential -- the Melnikov method provides information on the degree to which the frequency components of the excitation promote transitions effectively.

In Section 2 we review basic results on Melnikov functions and processes, and chaotic dynamics results pertaining to the escape problem. The remaining sections are devoted to applications.
where \( h(\tau) \), the ordinate of the homoclinic orbit in the \( x_1, x_2 \) plane can be obtained by integrating Eq. 1 with initial conditions at the saddle point. It can further be shown that: (1) if \( M(t) \) has simple zeros the stable and unstable manifolds intersect transversely, and (2) transverse intersections are a necessary condition for chaotic behavior and escapes. A cross-section through intersecting stable and unstable manifolds, which yields a so-called homoclinic tangle, is depicted in Fig. 4.

![Fig. 4. Plane cross-section through intersecting stable and unstable manifolds. Chaotic transport takes point \( A_2 \), located within the core bounded by the pseudoseparatrix, through \( A_1 \) and \( A_0 \), to point \( A_4 \), located outside the core.](image)

It can be shown that the homoclinic tangle makes possible the occurrence of chaotic motions and escapes of the type shown in Fig. 2c.

If \( G(t) \) is quasiperiodic (i.e., a sum of harmonic terms with amplitudes \( a_i \), generally incommensurate frequencies \( \omega_i \), and phase angles \( \theta_i \) \((i=1,2,\ldots,n)\)), it follows from Eqs. 2

\[
m(t) = \sum_{i=1}^{n} a_i |\alpha(\omega_i)| \cos[\omega_i t + \theta_i - \psi(\omega_i)]
\]

in which the transfer function \( \alpha(\omega_i) \) is the Fourier transform of \( h(\tau) \), and \( \psi(\omega_i) \) is the argument of \( \alpha(\omega_i) \) (Wiggins, 1988).

A physically realizable stochastic excitation can be closely approximated by a sum of harmonic terms with random parameters. A realization of the stochastic process corresponds to a fixed set of parameters. Therefore, instead of a Melnikov function, a system with a stochastic excitation possesses an ensemble of Melnikov functions, i.e., a Melnikov process. If the system’s excitation \( G(t) \) is Gaussian with unit variance, the Melnikov process has expectation \( -\beta c \) and spectral density

\[
\Psi_x(\omega) = \beta^2 |\alpha(\omega)|^2 \Psi_G(\omega),
\]

where \( \Psi_G(\omega) \) is the spectral density of \( G(t) \). Eq. 4 shows that, depending upon its shape, \( \alpha(\omega) \) may reduce or increase the contribution of the excitation’s various frequency components to the Melnikov process. This observation is useful in practical applications, notably the open-loop control of escapes.

3. CONDITION FOR NONOCCURRENCE OF ESCAPES IN SYSTEMS EXCITED BY DICHTOMOUS NOISE

The dichotomous noise \( G(t) \) we consider is depicted in Fig. 5. No escapes can occur if \( M(t) \) has no simple zeros. For example, since \( |G(t)| \leq 1 \), for the double-well potential \( V(x) = x^4/4 - x^2/2 \), which has the shape shown in Fig. 1a, the condition that Eq. 2 not have simple zeros (that escapes cannot occur) yields \( \gamma / \beta < 0.47 \) (Sivathanu, Hagwood, and Simiu, 1995). To our knowledge this is the only method for obtaining a criterion guaranteeing the nonoccurrence of escapes due to dichotomous noise.

4. VESSEL CAPSIZING

The empirical restoring force in the equation of rolling motion of a vessel is derived from an M-shaped potential, rather than a W-shaped potential as in Fig. 1a. The spectrum of the Melnikov process induced by the wave force has the form of Eq. 4; the spectrum of the wave force is equal to the spectrum of the waves times the square of the modulus of an empirical transfer function.

Using the phase space flux factor -- a functional of the Melnikov function, -- Hsieh, Troesch, and Shaw (1994) estimated capsizing probabilities of a vessel for various time intervals and significant wave heights.
5. OPEN-LOOP CONTROL OF ESCAPES

Consider Eq. 1, where $G(t) = G(t) - kG(t - \tau)$, $G(t)$ is a stationary excitation process, $kG(t - \tau)$ is a stationary control process whose addition to the system will reduce the escape rate, $\tau$ is a time lag, and $0 < k < 1$. For the trivial choice $G(t)/G(t - \tau)$ and small enough $\tau$ the escape rate will be smaller in the controlled than in the uncontrolled system. For $\tau = 0$ the ratio between the average powers of the control and excitation force is $q = k^2$. Melnikov theory can be used to obtain open-loop control forces that achieve reductions comparable to those due to a trivial control force, but with a smaller ratio $q$, i.e., with less energy. It follows from Eq. 4 that this can be accomplished if the frequency content of the control force is concentrated in the frequency interval where $|\alpha(\omega)|$ is highest. For details and results, see Simiu and Franaszek (1997) and Basios et al. (1999).

6. STOCHASTIC RESONANCE

Stochastic resonance is the phenomenon wherein, for a system excited by a low-frequency harmonic in the presence of noise, the output signal-to-noise ratio can be enhanced by adding noise or a harmonic excitation. From Melnikov theory it follows that the spectral density of the added noise or the frequency of the added harmonic excitation must correspond to the largest ordinates of $|\alpha(\omega)|$. The enhancement is due to the chaotic nature of the motion, similar to the motion of Fig. 2c, induced in the system by the excitation, the original noise, and the added noise or harmonic. If the escape rate is approximately equal to the excitation frequency, energy from the broadband spectrum inherent in the chaotic motion is transferred by a synchronization-like mechanism at that frequency, thus enhancing the signal-to-noise ratio (Franaszek, Simiu, 1996) – see Fig. 6.

7. SNAP-THROUGH OF A TRANSVERSELY EXCITED BUCKLED COLUMN

Holmes and Marsden (1981) applied Melnikov theory to a buckled column with continuous mass subjected to a continuously distributed transverse excitation with harmonic time dependence. This dynamical system can be shown to have restoring forces derived from a potential similar to Fig. 1a. An extension of the theory to the case of excitation by dichotomous noise yields a criterion for the nonoccurrence of snap-through similar to the criterion of Section 3 – see Simiu and Franszek (1996). An evolution in time of the buckled column shape is depicted in Fig. 7.

8. WIND-INDUCED ALONG-SHORE CURRENTS OVER OCEAN FLOOR WITH VARIABLE TOPOGRAPHY

Observations yielded by moored current meters suggest that wind-induced along-shore mesoscale velocity fields over a sloping, corrugated ocean floor (Fig. 8) contain energy in a continuous range of low frequencies, a possible indication of chaotic behavior. Allen et al. (1991) developed a simplified model of the flow to which the Melnikov approach is applicable. For wind speed fluctuations conforming to accepted meteorological models, Simiu (1996) obtained criteria for the probability that the flow motion can be chaotic during given time intervals.
9. THE AUDITORY NERVE SYSTEM AS A CHAOTIC DYNAMICAL SYSTEM

Figure 9 shows a typical time history of the response of a nerve fiber to harmonic excitation in the presence of weak noise. Experimental results reported in the 1960's are consistent with a model of the nerve fiber response, including irregular “firings,” as a planar dynamical system to which the Melnikov approach is applicable (Franaszek and Simiu, 1998). Similar models can be used to represent and analyze encephalograms and other plots of physiological or physical phenomena exhibiting irregular transitions.

REFERENCES


